



3.3 Examples of subspaces





When is a subset a subspace?



Introduction

- In this topic, we will
 - Revisit the subspaces of finite-dimensional vector spaces
 - Consider subspaces of discrete signals
 - Consider some subspaces of polynomials





The span of a set of vectors

- Given any vector space $\mathcal{V}$ and any finite set of vectors

$$\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$$

then all linear combinations of these vectors forms a subspace of $\mathcal{V}$

- We will give this subspace a special name:

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$$

- Thus, the span of a set of vectors is always a subspace of the vector space from which the vectors came





Polynomials

- We also saw that all polynomials of degree less than or equal to 5 also forms a subspace of the set of all polynomials $\mathbf{R}[x]$
 - Notice that this equals

$$\text{span} \left\{ \hat{1}, x, x^2, x^3, x^4, x^5 \right\}$$





Polynomials

- We saw that all vectors of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ do not form a subspace
- Question:
 - Do all vectors of the form $\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ define a subspace?
 - In this case, yes, because

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- One significant question we will be asking is when can we write one vector as a linear combination of other vectors?





Discrete signals

- The vector space of discrete signals includes, for example,

$$2^n = (1, 2, 4, 8, 16, 32, \dots)$$

which grows arbitrarily large

- We will define a discrete signal x as being *bounded* if there exists an $M > 0$ such that $|x[k]| \leq M$ for all $k = 0, 1, 2, \dots$





Discrete signals

- Question:
 - Is the set of all discrete signals bounded by 10 a subspace of all discrete signals?
 - For this to be true
 - Given such a signal x , then αx must also be bounded by 10
 - Given two such signals x and y , then $x + y$ must also be bounded by 10
 - Simple counter-example:
 - Consider the discrete signal $10 = (10, 10, 10, 10, 10, \dots)$
 - Each entry is $|10[k]| = 10 \leq 10$, so 10 falls into this set
 - However, $2 \cdot 10 = (20, 20, 20, 20, 20, \dots)$ and $|(2 \cdot 10)[k]| = 20 > 10$
 - Thus, it is not a subspace





Discrete signals

- Question:
 - Is the set of all bounded discrete signals a subspace?
 - Given x , there must be an M_x such that $|x[k]| \leq M_x$
 - Thus, given such an x , let us consider αx
 - Thus,
$$\begin{aligned} |(\alpha x)[k]| &= |\alpha x[k]| \\ &= |\alpha| |x[k]| \\ &\leq |\alpha| M_x \end{aligned}$$
 - Therefore, for x and y , there are M_x and M_y such that $|x[k]| \leq M_x$ and $|y[k]| \leq M_y$
 - Thus,
$$\begin{aligned} |(x+y)[k]| &= |x[k] + y[k]| \\ &\leq |x[k]| + |y[k]| \\ &\leq M_x + M_y \end{aligned}$$





Discrete signals

- Question:

- If $D^n \delta$ is defined such that

$$(D^n \delta)[k] = \begin{cases} 1 & k = n \\ 0 & \text{otherwise} \end{cases}$$

- Question: what is $\text{span}\{D^0 \delta, D^1 \delta, D^2 \delta, D^3 \delta, D^4 \delta, D^5 \delta\}$?

$$D^0 \delta = (1, 0, 0, 0, 0, 0, 0, \dots)$$

$$D^1 \delta = (0, 1, 0, 0, 0, 0, 0, \dots)$$

$$D^2 \delta = (0, 0, 1, 0, 0, 0, 0, \dots)$$

$$D^3 \delta = (0, 0, 0, 1, 0, 0, 0, \dots)$$

$$D^4 \delta = (0, 0, 0, 0, 1, 0, 0, \dots)$$

$$D^5 \delta = (0, 0, 0, 0, 0, 1, 0, \dots)$$

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 0, 0, \dots)$$

- Thus, the span of these six discrete signals is all signals that are zero if $k > 5$





Discrete signals

- Question:
 - A discrete signal x is *eventually zero* if there is a K such that $x[k] = 0$ whenever $k > K$
 - Is this a subspace of the vector space of all discrete signals?
 - Given x as above, if $k > K$ then $(\alpha x)[k] = \alpha x[k] = \alpha \cdot 0$
 - Thus αx is also eventually zero
 - Given x and y , if these discrete signals are eventually zero, there are K_x and K_y such that $x[k] = 0$ whenever $k > K_x$
 $y[k] = 0$ whenever $k > K_y$
 - Let $K = \max\{K_x, K_y\}$, and if $k > K$,
$$(x + y)[k] = x[k] + y[k] = 0 + 0 = 0$$
 - Thus, this is a subspace.





Polynomials

- Question:
 - Let r be a real number
 - Is the set of all polynomials that have a root at r a subspace of the vector space of all polynomials $\mathbf{F}[x]$?
 - If p is a polynomial with a root at r ,
then $(\alpha p)(r) = \alpha p(r) = \alpha \cdot 0 = 0$
 - If p and q are polynomials both with a root at r ,
then $(p + q)(r) = p(r) + q(r) = 0 + 0 = 0$
 - Thus, this is a subspace





Polynomials

- Question:
 - Let r be a real number
 - Is the set of all polynomials such that $p(r) = 1$ a subspace of the vector space of all polynomials $\mathbf{F}[x]$?
 - If p is a polynomial such that $p(r) = 1$,
then $(2p)(r) = 2p(r) = 2 \cdot 1 = 2$
 - Thus, this is not a subspace





Summary

- Following this topic, you now
 - Understand the idea of a subspace
 - Are aware of the span of a set of vectors
 - Have seen a number of subspaces of discrete signals
 - Have seen a number of subspaces of polynomials





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References

- [1] https://en.wikipedia.org/wiki/Linear_subspace





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Acknowledgments

None so far.





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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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